QUATERNIONS.

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Trigonometry.—HAMILTON does not give the quaternion equivalents for all the transformations of Trigonometry, and those which he does give are widely scattered through his works. I have found a *re sume'* of the subject fruitful in suggesting new relations between the symbols of quaternions.

The following formulæ from the *Elements* will be made use of:—

210. XI. $S.q^2 = (Sq)^2 + (Vq)^2.$ (1)

XII. $V.q^2 = 2Sq.Vq.$ (2)

XV. $Tq^2 = (Sq)^2 - (Vq)^2.$ (3)

XVII. $SU(q)^2 = 2(SUq)^2 - 1.$ (4)

XVIII. $Sq'q = Sq'.Sq + S(Vq'.Vq).$ (5)

XIX. $Vq'q = Vq'.Sq + Vq.Sq' + V(Vq'.Vq).$ (6)

274. XII. $(TV{:}S)\sqrt{q} = \sqrt{[(Tq - Sq) \div (Tq + Sq)]}.$ (7)

XI. $TV\sqrt{q} = \sqrt{\tfrac{1}{2}(Tq - Sq)}.$ (8)

199. XII. $S\sqrt{q} = \sqrt{\tfrac{1}{2}(Tq + Sq)}.$ (9)

205. $V\Sigma = \Sigma V,\ S\Sigma = \Sigma S.$ (10)

294. II. $V.\gamma V\beta\alpha = \alpha S\beta\gamma - \beta S\gamma\alpha.$ (11)

210. XXIX. $Tq' + Tq = T(q' + q)$ if $q' = xq$. (12)

$S.\gamma V\beta\alpha = \gamma\beta\alpha - \gamma S\beta\alpha + \beta S\gamma\alpha - \alpha S\beta\gamma.$ (13)

We have, for the trigonometrical functions,

$$\sin\ \angle q = TVUq, \qquad \cos\ \angle q = SUq,$$
$$\sin n\angle q = TVU(q^n), \qquad \cos n\angle q = SU(q^n),$$
$$\tan\ \angle q = (TV{:}S)q.$$

Putting Uq for q in (1) and taking the tensor,

$$SU(q^2) = (SUq)^2 + (TVUq)^2, \tag{20}$$

$\cos 2x = \cos^2 x - \sin^2 x$; or from (4)

$$\cos 2x = 2\cos^2 x - 1.$$

By treating (2) in a similar way we have

$$\sin 2x = 2\sin x \cos x.$$

From (3) $$1 = (SUq)^2 - (VUq)^2, \tag{21}$$

or $$1 = \sin^2 x + \cos^2 x.$$

Introducing in (6) the condition that q' and q be complanar, and substituting versors, we have $VUq'q = VUq'SUq + VUq\,SUq'$.

Taking the tensor of this equation and observing that $VUq' \parallel VUq$, we have, by (12), $$TVUq'q = TVUq'\,SUq + SUq'\,TVUq \tag{22}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

We have from (5), since

$$S(Vq'.Vq) = -TVq'TVq\cos\angle(Vq':Vq) = -TVq'.TVq,$$

$$SUq'q = SUq'SUq - TVUq'TVUq \tag{23}$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y.$$

Substituting q^{-1} for q in the last two equations,

$$TVUq'q^{-1} = TVUq'SUq - SUq'TVUq, \tag{24}$$

$$SUq'q^{-1} = SUq'SUq + TVUq'TVUq, \tag{25}$$

which gives the values for the sine and cosine of the difference of two angles.

Adding (22) and (24), $TVUq'q + TVUq'q^{-1} = 2SUq\,TVUq'S.$

Putting $q'q^{-1} = r$, $q'q = r'$, $q' = \sqrt{(r'r)}$, $q = \sqrt{(rr')}$ and we have

$$TVUr' + TVUr = 2SU\sqrt{(r'r^{-1})}TVU\sqrt{(r'r)},$$

$$\sin x + \sin y = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y).$$

Similarly,

$$TVUr' - TVUr = 2SU\sqrt{(r'r)}TVU\sqrt{(r'r^{-1})},$$

$$SUr' + SUr = 2SU\sqrt{(r'r)}SU\sqrt{(r'r^{-1})},$$

$$SUr' - SUr = -2TVU\sqrt{(r'r)}\,TVU\sqrt{(r'r^{-1})}.$$

From (8) $\qquad 2(TVU\sqrt{q})^2 = TUq - SUq,$

$$2\sin^2\tfrac{1}{2}x = 1 - \cos x.$$

From (9) $\qquad 2(SU\sqrt{q})^2 = TUq + SUq,$

$$2\cos^2\tfrac{1}{2}x = 1 + \cos x.$$

$$(TV{:}S)q^2 = \frac{2Sq.TVq}{(Sq)^2 + (Vq)^2} = \frac{2TVq}{Sq}\cdot\frac{(Sq)^2}{(Sq)^2 - (TVq)^2} = \frac{2(TV{:}S)q}{1-[(TV{:}S)q]^2},$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}.$$

$$(TV{:}S)q'q = \frac{(TV{:}S)q+(TV{:}S)q'}{1-(TV{:}S)q(TV{:}S)q'},\quad (TV{:}S)q'q^{-1} = \frac{(TV{:}S)q-(TV{:}S)q'}{1+(TV{:}S)q(TV{:}S)q'},$$

$$\tan(x\pm y) = \frac{\tan x \pm \tan y}{1\mp\tan x\tan y}.$$

$$(TV{:}S)q' + (TV{:}S)q = \frac{TVUq'q}{SUq'\,SUq},\quad (TV{:}S)q' - (TV{:}S)q = \frac{TVUq'q^{-1}}{SUq'\,SUq},$$

$$\tan x \pm \tan y = \frac{\sin(x\pm y)}{\cos x\cos y}.$$

$$(TV{:}S)\sqrt{(r'r)} = \frac{TVUr' + TVUr}{SUr' + SUr},\quad (TV{:}S)\sqrt{(r'r^{-1})} = \frac{TVUr' - TVUr}{SUr' + SUr},$$

$$\tan\tfrac{1}{2}(x\pm y) = \frac{\sin x \pm \sin y}{\cos x + \cos y}.$$

By (6) we have

$$\begin{aligned} V(q''.q'q) &= Vq'qSq'' + Sq'q\,Vq'' + V(Vq''.Vq'q) \\ &= Vq'Sq\,Sq'' + Vq\,Sq'Sq'' + Sq''\,V.Vq'\,Vq + Vq''Sq'Sq \\ &\quad + Vq''S.Vq'\,Vq + V(Vq''\,Vq'Sq + Vq''\,VqSq' + Vq''\,V.Vq'\,Vq), \end{aligned}$$

which becomes, by (10) and (11) and by arranging symmetrically,

$$Vq''q'q = Vq\,Sq'\,Sq'' + Vq'Sq''Sq + Vq''Sq'Sq + Sq\,V.Vq''\,Vq' - Sq'\,V.Vq\;Vq'' + Sq''\,V.Vq'\,Vq + VqS.Vq''\,Vq' - Vq'S.Vq\;Vq'' + Vq''S.Vq'\,Vq. \tag{39}$$

By (5) $S(q''q'q) = Sq''Sq'q + S(Vq''.Vq'q) = Sq''Sq'Sq + Sq''S.Vq'\,Vq + S[Vq''\,Vq'Sq + Vq''\,Vq\,Sq' + Vq''\,V(Vq'\,Vq)]$, or by (10) and (13),

$$S.q''q'q = Sq''Sq'Sq + Vq''\,Vq'\,Vq + Sq''S.Vq'\,Vq - Vq''S.Vq'\,Vq + Sq'S.Vq\,Vq'' + Vq'S.Vq\,Vq'' + Sq\,S.Vq''\,Vq' - Vq\,S.Vq'\,Vq'$$
$$= Sq''Sq'Sq + Vq''\,Vq'\,Vq + q'S.Vq\,Vq'' + KqS.Vq''\,Vq' + Kq''S.Vq'\,Vq. \tag{40}$$

Making the quaternions complanar and their tensors equal, and then taking the tensors of the two equations, we have for the sine and cosine of the sum of three angles,

$$TV\Upsilon^* q''q'q = TV\Upsilon q''S\Upsilon q'S\Upsilon q + S\Upsilon q''TV\Upsilon q'S\Upsilon q + S\Upsilon q''S\Upsilon q'TV\Upsilon q - TV\Upsilon q''TV\Upsilon q'TV\Upsilon q. \tag{41}$$

$$S\Upsilon q''q'q = S\Upsilon q''S\Upsilon q'S\Upsilon q - TV\Upsilon q''TV\Upsilon q'S\Upsilon q - TV\Upsilon q''S\Upsilon q'TV\Upsilon q - S\Upsilon q''TV\Upsilon q'TV\Upsilon q. \tag{42}$$

These equations might have been deduced directly from (22) and (23), but the formulæ for non-complanar quaternions, (39) and (40), are not without value.

From (22), (23), (20), (2), (41) and (42), we obtain

$$TV\Upsilon q''^2 + TV\Upsilon q'^2 + TV\Upsilon q^2 - TV\Upsilon q''^2q'^2q^2 = 4TV\Upsilon q''q'\,TV\Upsilon q'q\,TV\Upsilon qq''; \tag{43}$$

$$S\Upsilon q''^2 + S\Upsilon q'^2 + S\Upsilon q^2 + S\Upsilon q''^2q'^2q^2 = 4S\Upsilon q''q'\,S\Upsilon q'q\,S\Upsilon qq''; \tag{44}$$

$$-TV\Upsilon q''^2 + TV\Upsilon q'^2 + TV\Upsilon q^2 + TV\Upsilon q''^2q'^2q^2 = 4S\Upsilon q''q'\,TV\Upsilon q'q\,S\Upsilon qq''; \tag{45}$$

$$-S\Upsilon q''^2 + S\Upsilon q'^2 + S\Upsilon q^2 - S\Upsilon q''^2q'^2q^2 = 4TV\Upsilon q''q'\,S\Upsilon q'q\,TV\Upsilon qq''. \tag{46}$$

When $\angle q''q'q = 2n.\frac{1}{2}\pi$, $TV\Upsilon q''^2q'^2q^2 = 0$ and $S\Upsilon q''^2q'^2q^2 = -1$.

The above equations then become

$$\pm 4TV\Upsilon q\;TV\Upsilon q'\;TV\Upsilon q'' = TV\Upsilon q^2 + TV\Upsilon q'^2 + TV\Upsilon q''^2,$$
$$\pm 4S\Upsilon q\;S\Upsilon q'\;S\Upsilon q'' = S\Upsilon q^2 + S\Upsilon q'^2 + S\Upsilon q''^2 + 1,$$
$$\pm 4S\Upsilon q\;S\Upsilon q'\;TV\Upsilon q'' = TV\Upsilon q^2 + TV\Upsilon q'^2 - TV\Upsilon q''^2,$$
$$\pm 4TV\Upsilon q\;TV\Upsilon q'\;S\Upsilon q'' = S\Upsilon q^2 + S\Upsilon q'^2 - S\Upsilon q''^2 - 1,$$

the upper sign being taken when n is even, the lower when n is odd.

When $\angle q''q'q = (2n+1)\frac{1}{2}\pi$, $TV\Upsilon q''^2q'^2q^2 = 0$ and $S\Upsilon q''^2q'^2q^2 = 1$, and the equations become

$$\pm 4S\Upsilon q\;S\Upsilon q'\;S\Upsilon q'' = TV\Upsilon q^2 + TV\Upsilon q'^2 + TV\Upsilon q''^2,$$
$$\pm 4TV\Upsilon q\;TV\Upsilon q'\;TV\Upsilon q'' = S\Upsilon q^2 + S\Upsilon q'^2 + S\Upsilon q''^2 - 1,$$
$$\pm 4TV\Upsilon q\;TV\Upsilon q'\;S\Upsilon q'' = TV\Upsilon q^2 + TV\Upsilon q'^2 - TV\Upsilon q''^2,$$
$$\pm 4S\Upsilon q\;S\Upsilon q'\;TV\Upsilon q'' = S\Upsilon q^2 + S\Upsilon q'^2 - S\Upsilon q''^2 + 1,$$

according as n is even or odd.

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*For want of sorts the Greek Upsilon is here used instead of U.—Compositor.